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Optimal reactions for the search for $\pi^0 \to e^+e^-$

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Abstract. We examine various reactions in which one might search for $\pi^0 \to e^+e^-$; apart from the problem of the low branching ratio, the most serious observational difficulties arise from the competition of intermediate virtual photon states of close to the pion mass. We show that the obvious choice of $\pi^-p \to \pi^0n$ at threshold is an unfavourable pion source but that the same reaction in the $\Delta(1236)$ resonance region seems currently to offer the best opportunity; we analyse it in detail. $K^+ \to \pi^+\pi^0$ seems the most hopeful alternative.

1. Introduction

Rare leptonic decay modes of the neutral pseudoscalar mesons are of great interest because of the information they may reveal about neutral currents (see eg Marshak *et al* 1969) and, more recently, the possibility of new CP violating channels of K decay (Clark *et al* 1971, Carithers *et al* 1973, Gjesdal *et al* 1973). Much experimental evidence has been compiled for $K \to l^+l^-$ (Clark *et al* 1971, Carithers *et al* 1973, Gjesdal *et al* 1973, see Particle Data Group 1973 for a critical comparison and earlier references) and there have been two experiments on $\eta \to \mu^+\mu^-$ (Hyams *et al* 1969, Wehmann *et al* 1968). However, no one has observed $\pi \to e^+e^-$.

From angular momentum, C and P conservation alone, the rate for direct lepton-pair decays of 0^- particles can be shown to be proportional to the square of the lepton mass. As a result, electron–positron pair decays are considerably more difficult to observe than the muon mode. Since $\pi^0 \to \mu^+\mu^-$ is impossible, the lepton-pair mode of pion decay is particularly rare, so a high π^0 flux is required. To reduce background, an experiment should also identify and momentum analyse the lepton pair so as to reconstruct the π^0 mass. Kinematically indistinguishable background can arise from other π^0 decay modes, eg $\pi^0 \to \gamma e^+e^-$ or from processes competing with π^0 production, particularly photoemission. Section 2 shows that the former is negligible, evaluates the latter for various incident beams and selects the optimal reaction among those considered. Section 3 analyses this reaction in detail.

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2. General survey

2.1. $\pi^0 \to e^+e^-$

This decay had been studied by several authors (Drell 1959, Berman and Geffen 1960, Young 1967, Quigg and Jackson 1968, Litskevitch and Franke 1970, Litskevitch 1970, Pratap and Smith 1972). Here we only wish to remark that the branching ratio

$$B = \frac{\Gamma(\pi^0 \to e^+e^-)}{\Gamma(\pi^0 \to 2\gamma)} \tag{1}$$

will be assumed to have the approximate value 6×10^{-8}.

There exists an unambiguous, model-independent lower bound for B of $4{\cdot}7 \times 10^{-8}$. The imaginary part of the amplitude from unitarity for the 2γ intermediate state is responsible for this bound. On the other hand, for the real part, the $\pi\gamma\gamma$ vertex function with off-shell photons is needed. The logarithmically divergent result, which comes from assuming the vertex function to be a constant, may be made finite by constructing reasonable models. Until recently, values for B, which are insensitive to model parameters, group around 6×10^{-8} (Quigg and Jackson 1968). The latest model (Pratap and Smith 1972) predicts $B = 14 \times 10^{-8}$. For definiteness, we will assume the former value in this paper. If experimental evidence were to indicate a branching ratio much higher than 10^{-7}, then neutral currents will have to be postulated. If, on the other hand, B were to be below $4{\cdot}7 \times 10^{-8}$, then *CP violating* neutral currents would be needed, since any CP conserving fundamental interaction of $e^+e^-\pi^0$ must have a coupling which gives a real Born approximation and thus no interference with the unitarity contribution.

2.2. *Background of single Dalitz pairs ($\pi^0 \to \gamma e^+e^-$)*

This decay has been observed, with a branching ratio (Particle Data Group 1973) of

$$B_\gamma = (1{\cdot}17 \pm 0{\cdot}04) \times 10^{-2}, \tag{2}$$

which is considerably higher than B. When the photon is soft this process poses as a background to the pure leptonic mode. Fortunately, the major contribution to (2) comes from high energy photons. Assuming the $\pi\gamma\gamma$ vertex to be independent of the mass of the virtual photon, an estimate for this background can be obtained (Drell 1959, Berman and Geffen 1960, Young 1967, Quigg and Jackson 1968, Litskevitch and Franke 1970, Litskevitch 1970). Integrating soft photon energies up to E_γ we have

$$B_{\gamma,E_\gamma} = \frac{8\alpha}{3\pi}\left(\frac{E_\gamma}{\mu}\right)^4; \qquad \mu = \pi^0 \text{ mass}.$$

If we demand that the invariant mass squared of the lepton pair lie within $\frac{1}{2}\delta\mu^2$ of μ^2, then

$$B_{\gamma,\delta} = \frac{\alpha}{96\pi}\delta^4, \tag{3}$$

with mass resolutions δ of a few per cent, we can safely neglect the background due to this process.

2.3. *Competition from an intermediate virtual photon (interval conversion)*

Throughout the following discussion, we shall assume that the spins of the leptons are not detected; otherwise, the leptons from pion decay, having the same helicity would be distinguishable from those from a virtual photon.

For each arbitrary process in which a neutral pion is created

$$\mathrm{A} \to \mathrm{B} + \pi^0, \tag{4a}$$

an accompanying virtual photon production process

$$\mathrm{A} \to \tilde{\mathrm{B}} + \text{“}\gamma\text{”} \tag{4b}$$

can take place. “γ” denotes a photon with the mass of the pion. The relevant quantum numbers of π^0 and “γ” are the same except for charge conjugation, parity and spin. In an experiment in which the details of the final states are not analysed, the two processes are directly competing since B and $\tilde{\mathrm{B}}$ can be the same particle(s).

Of course, the cross section for the latter process is smaller, typically, by a factor of α. But, on the other hand, the branching ratio of the lepton pair mode of the pion decay is of the order of $\alpha^2(M_e/\mu)^2$†, whereas “γ” $\to \mathrm{e}^+\mathrm{e}^-$ is of order α. The other helpful factor is that only heavy photons of the correct mass range can compete so that only a small fraction (order δ) of the total internal conversion process ($\mathrm{A} \to \tilde{\mathrm{B}} + \mathrm{e}^+\mathrm{e}^-$) poses as background. A rough order of magnitude estimate of the signal to noise ratio runs as follows.

For the pion case, we have, exactly

$$\sigma(\mathrm{A} \to \mathrm{B} + \mathrm{e}^+\mathrm{e}^-) = \sigma(\mathrm{A} \to \mathrm{B} + \pi^0)B. \tag{6}$$

For the “γ” case, we estimate

$$\sigma(\mathrm{A} \to \tilde{\mathrm{B}} + \mathrm{e}^+\mathrm{e}^-) \sim \sigma(\mathrm{A} \to \tilde{\mathrm{B}} + \gamma)\rho_\delta \tag{7}$$

where ρ_δ is the internal conversion coefficient for electron–positron pairs of square invariant mass μ^2 and resolution $\delta\mu^2$. Using the general formula of Kroll and Wada (1955), we have

$$\rho_\delta \sim \frac{\alpha\delta}{3\pi}\frac{k_\pi}{k_\gamma}\left(R_\mathrm{T} + \frac{\mu^2}{2E_\pi^2}R_\mathrm{L}\right), \tag{8}$$

where $k_{\pi(\gamma)}$ is the momentum of a real pion (photon) in the rest frame of A, $E_\pi^2 = k_\pi^2 + \mu^2$ and $R_{\mathrm{T(L)}}$ is the ratio of the transition probability of an off-shell transverse (longitudinal) photon to that of a real one. Terms of order $\mu^2/(M_\mathrm{A}^2 + M_\mathrm{B}^2)$ and M_e^2/μ^2 compared with unity have been neglected, since this is true for the cases we are considering.

Combining (6), (7) and (8), we have

$$R \equiv \frac{\sigma(\mathrm{A} \to \mathrm{B} + \mathrm{e}^+\mathrm{e}^-)}{\sigma(\mathrm{A} \to \tilde{\mathrm{B}} + \mathrm{e}^+\mathrm{e}^-)} \sim BR_\sigma\left(\frac{3\pi k_\gamma}{\alpha\delta k_\pi}\right)\left(R_\mathrm{T} + \frac{\mu^2}{2E_\pi^2}R_\mathrm{L}\right)^{-1} \tag{9}$$

where

$$R_\sigma \equiv \frac{\sigma(\mathrm{A} \to \mathrm{B} + \pi^0)}{\sigma(\mathrm{A} \to \tilde{\mathrm{B}} + \gamma)}. \tag{10}$$

† M_e is the electron mass.

Let us consider a few examples with δ being 1 %.

(i) $\pi^- p \to \pi^0 n$ at rest.

The idea of producing π^0 this way has several attractive features: available high π fluxes are efficiently converted; the lepton pair system is monoenergetic and their tracks are practically co-linear. However, because the π–N scattering lengths are small, the internal conversion rate is relatively large.

Estimates of R_T and R_L exist for this threshold reaction (Kroll and Wada 1955): $R_T \sim 1$, $R_L \sim 2$. R_σ, the Panofsky ratio, is 1·5. k_γ/k_π is about 5. Thus,

$$R \sim \tfrac{1}{30}. \tag{11}$$

Such a low value of signal to noise ratio renders this reaction extremely unfavourable for the search for the rare leptonic mode.

(ii) $\pi^- p \to \pi^0 n$ in the $\Delta(1236)$ region.

In the region of the 33 resonance, R_σ rises to about 75 (Dance 1971)† while $k_\gamma/k_\pi \sim 1$. Table 1 shows the value $R_\sigma(k_\gamma/k_\pi)$ for various initial pion laboratory kinetic energies.

On the other hand, electroproduction data (Clegg 1969) show that the longitudinal cross sections are typically small compared to the transverse ones. We will assume that this is the case in the time-like virtual photon region as well and keep in mind that, by setting the last factor in expression (9) to unity, we may be making errors of the order of 10–15 %. The result is

$$R \sim 0{\cdot}6 \tag{12}$$

with 170 MeV $< T_\pi <$ 230 MeV. Although this value is still rather low, it is acceptable; we shall see later that it can be improved by making use of the difference in angular distributions of scattering and photoproduction.

Table 1. Cross sections and their ratio

T_π (MeV)	$\sigma(\pi^- p \to \pi^0 n)$ (mb)	$\sigma(\pi^- p \to \gamma n)$ (mb)	$R_\sigma(k_\gamma/k_\pi)$
90·9 ± 0·4	15·36 ± 0·18	0·67	30·1
115·3 ± 0·5	25·01 ± 0·23	0·71	43·8
143·3 ± 0·5	40·13 ± 0·27	0·76	62·9
163·8 ± 0·6	47·89 ± 0·32	0·71	78·5
168·1 ± 0·6	48·44 ± 0·35	0·70	80·3
177·1 ± 0·7	48·13 ± 0·30	0·66	83·8
192·3 ± 0·7	44·85 ± 0·45	0·60	84·7
215·0 ± 0·7	36·61 ± 0·22	0·48	85·3
236·9 ± 0·8	29·26 ± 0·20	0·41	78·9
260·6 ± 0·8	23·45 ± 0·21	0·35	73·2
291·6 ± 0·9	17·91 ± 0·17	0·31	62·5

We may also think of this reaction as first producing a Δ and then study its decay into πN against γN. Looking at the branching ratios of Δ, we can also arrive at $R \sim O(1)$. However, the charge exchange and the photo-emission processes are not completely dominated by the Δ resonance and a more detailed analysis is needed. Section 3 is devoted to this.

(iii) $\gamma p \to \pi p$: $pp \to \pi pp$.

To ensure a large cross section for producing pions, we again consider the Δ region.

† See also *Nuclear Physics* B **26** 588.

We base the analysis on the branching ratios in Δ decay and again obtain $R \sim O(1)$; the absence of detailed information on the photon-producing sister processes makes an accurate prediction more difficult here.
(iv) $K^+ \to \pi^+\pi^0$.
Since kaons and pions both have zero spin, there is no real photo-emission ($K^+ \not\to \pi^+\gamma$). However, internal conversion $K \to \pi e^+e^-$ can occur (Dalitz 1955, Cabibbo and Ferrari 1960, Baker and Glashow 1962). The best way to estimate R in this case is (*a*) to assume that the shape of the decay spectrum of $\pi^+e^+e^-$ is the same as $\pi^0 e\nu$ (Beg 1963), (*b*) normalize the total branching ratio of this mode to the present experimental upper limit 4×10^{-7} (Graham and Pakavasa 1965), and (*c*) find the fraction of the decay rate with the appropriate invariant mass of the lepton pair. This procedure leads to

$$R \gtrsim O(10).$$

(v) $\Sigma^+ \to \pi^0 p$.
In this case, the branching ratio for $\pi^0 p$ is about half, as opposed to the $p\gamma$ mode of 10^{-3}. Thus $R \sim 500$ giving also

$$R \sim O(10).$$

Here, we have again assumed that R_L is small to give this order of magnitude estimate.

2.4. Discussion

There are of course many other ways of producing pions than the few examples given above. However, they serve to illustrate the two major limitations to be faced in the search for $\pi^0 \to e^+e^-$: background from internal conversion and rate limitations from the π^0 flux. These examples were listed in the order of increasing R but, it also happens, of roughly decreasing π^0 flux.

If the only concern is a large value of R, we might consider processes such as $\Xi^0 \to \Lambda\pi^0$, where theoretical expectations (Graham and Pakavasa 1965) of the radiative branching ratio are about 10^{-4}†. However, the rate is also important, and hyperon beams of the intensity required to investigate 10^{-8} branching ratios are not now, if ever, planned. $K^+_{\pi2}$ decay is potentially the simplest candidate but this experiment seems to require an improvement on currently available K^+ beams; $10^5 K^+ s^{-1}$ give a total of only about four $\pi^0 \to e^+e^-$ decays per hour into the full solid angle.

At the other extreme where π^0's are produced abundantly, the internal conversion background tends to be high. Limited principally by Coulomb scattering, it is improbable that the momentum resolution can be improved sufficiently to achieve an acceptably low background for $\pi^-p \to \pi^0 n$ at rest‡. In the vicinity of the Δ, charge-exchange cross sections are sufficiently large to give a useful source of π^0; R_σ is acceptably high and the π^0 mass resolution need not change significantly. The Δ can be produced with other intense beams; however $\gamma p \to \Delta \to e^+e^-p$ has a very large general background while the final state of $pp \to \Delta p \to e^+e^-pp$ makes experiment and analysis exceedingly complex.

On the basis of these remarks, we believe $\pi^-p \to \Delta \to e^+e^-n$ is at present the optimal reaction for searching for $\pi^0 \to e^+e^-$. In the following section, we will analyse this reaction

† Actually, $\Xi \to \Sigma^0\gamma$ can occur with branching ratio of the order of 10^{-3}.
‡ If Pratap and Smith (1972) were correct *and* if the π^0 mass resolution could be improved to 0·2%, the signal to noise ratio at threshold is still 1:3. By contrast, similar assumptions would yield a ratio of 6:1 for the charge-exchange reaction at the Δ.

in detail and show that the effective R can be increased from 0·6 by exploiting the angular distribution differences.

3. Detailed analysis of the optimal reaction

In this section, we carry out a detailed analysis for the signal to noise ratio for the reaction $\pi^- p \to \pi^0 n$ in the $\Delta(1236)$ region. We have seen that this ratio is about 0·6, but the angular distributions of the competing processes are quite different, so that we can exploit these to our advantage. We will find the best kinematical region and show that the ratio here is enhanced by a factor of about 10. The last section will indicate the kind of experimental configuration which favours this kinematical region.

Consider the CM angular distribution of the reaction $\pi^- p \to \pi^0 n$. If there were no background to the Δ resonance, the angular distribution would be $\frac{1}{2}(1+3z^2)$, where z is the cosine of the angle between the π^0 and π^- momenta. Experimentally, for $T_\pi \sim 190$ MeV ± 20 MeV, the angular distribution is well approximated by this (Källen 1964). At 180 MeV, for example, it is (cf figure 1)

$$\tfrac{1}{2}(1+3z^2-0{\cdot}3z) \tag{13}$$

again normalized to 4π. The lepton pair distribution is isotropic in the π^0 rest frame so that the distribution in the Δ rest frame is easily obtainable.

For the intermediate photon case, however, the situation is somewhat more involved. From photoproduction data (ABBHHM collaboration 1968), it is known that the two dominant contributions come from the magnetic dipole produced by the Δ and an electric dipole background. The resulting angular distribution is of the form

$$\frac{1}{4+2|a|^2}(5-3z^2+2|a|^2+4\,\mathrm{Re}\,az),$$

where a is the complex ratio of the electric to the magnetic contributions. Corresponding

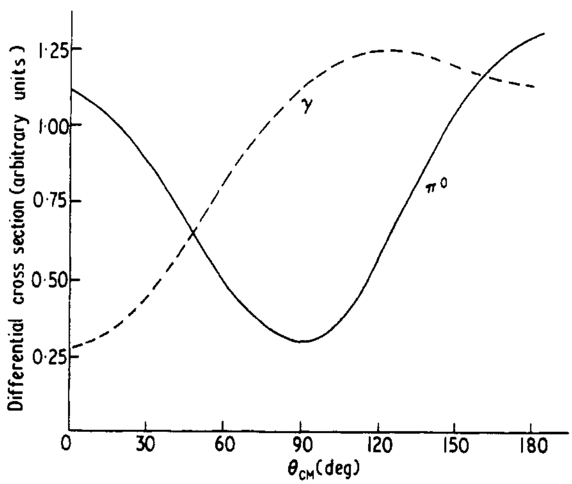

Figure 1. The angular distribution of photo-emission ($\pi^- p \to \gamma n$) and charge-exchange ($\pi^- p \to \pi^0 n$) processes at $T_\pi = 180$ MeV.

to the energy ($T_\pi \sim 180$ MeV) of (13), $\mathrm{Re}\, a \sim -\frac{3}{4}$ and $|a|^2 \sim \frac{3}{2}$ so that the photon distribution becomes, approximately (cf figure 1),

$$\tfrac{1}{7}(8-3z^2-3z). \tag{14}$$

Comparing (13) with (14), we see that the ratio of neutral pions to photons emerging in the forward direction is the most favourable. In general, in the region just below the Δ mass ($T_\pi < 190$ MeV, $E_\gamma < 340$ MeV), the forward–backward asymmetry of the photon angular distribution, together with the forward peaked pion distribution, gives us the best advantage.

In figure 1, we plot (13) and (14) with a slight variation. The units are arbitrary and the photon distribution is normalized so that the total is 4π. However, the pion normalization is $4\pi(0{\cdot}6)$, in accordance with (12). As a result, the figure represents the number of pions (or heavy photons of mass $\mu(1 \pm \frac{1}{2}\delta)$) decaying into e^+e^- against the centre of mass angle in the reaction $\pi^- p \to \pi^0(\gamma)n \to e^+e^-n$.

Proceeding on to the analysis of the lepton pair formation, we make use of the standard formalism of photoproduction (see eg Donnachie 1971 p 109). As we have noted before, we will neglect the scalar-longitudinal contributions, remembering that corrections are of the order of 10%–20%.

The angular distribution in the 'Δ rest frame' is

$$\mathscr{F}_\gamma = \frac{3}{8+4|a|^2} N\{5-3z^2+2|a|^2+4\,\mathrm{Re}\, az-(1+|a|^2+2\,\mathrm{Re}\, az)(\hat{\boldsymbol{d}}\times\hat{\boldsymbol{k}})^2-3[\hat{\boldsymbol{q}}\cdot(\hat{\boldsymbol{d}}\times\hat{\boldsymbol{k}})]^2\}, \tag{15}$$

where

$$\begin{aligned} \hat{\boldsymbol{q}} &= \boldsymbol{q}/|q|; & \boldsymbol{q} &= \pi^0 \text{ momentum} \\ \hat{\boldsymbol{k}} &= \boldsymbol{k}/|k|; & \boldsymbol{k} &= \boldsymbol{p}_+ + \boldsymbol{p}_- = \text{sum of lepton momenta} \\ \hat{\boldsymbol{d}} &= \boldsymbol{d}/\mu; & \boldsymbol{d} &= \boldsymbol{p}_+ - \boldsymbol{p}_- = \text{difference of lepton momenta} \\ z &= \hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}} \\ N &= \frac{2\sqrt{s}}{\pi^2|\boldsymbol{k}|}. \end{aligned}$$

The factor N serves to normalize the angular distribution so that integration over the three-particle final state momenta, subject to the condition $\delta(k_\mu^2-\mu^2)$, is unity.

The expression corresponding to (15) for the charge-exchange process is independent of $\hat{\boldsymbol{d}}\times\hat{\boldsymbol{k}}$. Using the *same normalization* as (15), it is

$$\mathscr{F}_\pi = \tfrac{1}{2}N(1-0{\cdot}32+3z^2). \tag{16}$$

Our object is to find the position and the value of the maximum of $\mathscr{F}_\pi/\mathscr{F}_\gamma$. Deferring the details to the appendix, we simply quote the result: at $T_\pi \simeq 180$ MeV, where $\mathrm{Re}\, a \sim -0{\cdot}75$ and $|a|^2 = 1{\cdot}5$,

$$\max(\mathscr{F}_\pi/\mathscr{F}_\gamma) \sim 10 \tag{17}$$

at

$$z = 1 \qquad \text{and} \qquad |\hat{\boldsymbol{d}}\times\hat{\boldsymbol{k}}| = 1. \tag{18}$$

Expression (17) shows that the signal to noise ratio can be effectively raised to 6 (from 0·6) if we restrict ourselves to the kinematical points specified by condition (18).

The physical situation represented by (18) is that the sum of the lepton momenta is (*a*) in the forward direction and (*b*) perpendicular to the difference of these momenta. We remark that, in this kinematical region, the leptons have identical energies. Finally we note that this situation remains the same in the laboratory frame, which is an added advantage.

4. Discussion and conclusions

In this paper, we have been concerned with the kinematically indistinguishable background in the search for the decay $\pi^0 \to e^+e^-$. With invariant mass-squared resolution of 1 % and an assumed branching ratio of 6×10^{-8}, we have shown that the background arising from the internal conversion process puts severe limits on the type of reaction used for the production of the neutral pion.

In particular, it is extremely unfavourable to use the charge-exchange reaction *at rest* for producing π^0. The internal conversion background ($\pi^-p \to$ "γ"$n \to e^+e^-n$) turns out to be approximately thirty times larger than the signal ($\pi^-p \to \pi^0 n \to e^+e^-n$). At the opposite end of the spectrum of reactions we surveyed, production of π^0 from certain strange particle decays leads to much larger signal to noise ratios (order of 10). Of these, $K^+ \to \pi^+ + \pi^0 \to \pi^+ + e^+ + e^-$ seems potentially to be the best, however, branching ratios of the order of 10^{-8} in hyperon or K^+ decays seem to be difficult at present because of intensity limits.

Given the present experimental technology, we believe that $\pi^-p \to \Delta(1236) \to \pi^0 n$ is the optimal reaction for searching for the rare leptonic mode of π^0 decay. Although the background is of the same order as the signal, the former can be reduced further by exploiting the difference in angular distributions between the photo-emission and charge-exchange processes. In the energy region just below the $\Delta(T_\pi \lesssim 190\,\text{MeV})$; there is a forward peak in the π^0 distribution where the "γ" distribution has a minimum. Moreover, in the kinematical region where the leptons have identical kinetic energies (in the laboratory frame or Δ rest frame), the background is further suppressed. The final signal to noise ratio is about 6, within 1 % mass resolution on the lepton pair.

Finally we estimate that, with a 180 MeV beam incident on a straightforward target configuration, $10^7\pi^-\ s^{-1}$ would give approximately 50 events per day. To keep the background lepton noise level low, an experiment would further require a low energy initial proton beam. Since a flux of this order is predicted in several places, measuring the rate of $\pi^0 \to e^+e^-$ seems feasible in the near future.

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Appendix

To find the maximum of $\mathscr{F}_\pi/\mathscr{F}_\gamma$, we will assume, for definiteness that $\mathrm{Re}\, a = 0{\cdot}75$ and $|a|^2 = 1{\cdot}5$. From (15), we have

$$\mathscr{F}_\gamma = \tfrac{3}{14}N\{8 - 3z - 3z^2 - \tfrac{1}{2}(5-3z)(\hat{\boldsymbol{d}}\times\hat{\boldsymbol{k}})^2 - 3[\hat{\boldsymbol{q}}\cdot(\hat{\boldsymbol{d}}\times\hat{\boldsymbol{k}})]^2\} \tag{A.1}$$

and, from (16),

$$\mathscr{F}_\pi = \tfrac{1}{2}N(1-0{\cdot}3z+3z^2). \tag{A.2}$$

As it turns out, a local maximum of (A.2) coincides with the minimum of (A.1). The former is just

$$(\mathscr{F}_\pi)_{\text{local max}} \sim (1{\cdot}85)N \tag{A.3}$$

at

$$z = 1. \tag{A.4}$$

Note that (A.4) is independent of $\hat{\boldsymbol{d}}\times\hat{\boldsymbol{k}}$.

Turning to (A.1), we remark that $|\hat{\boldsymbol{d}}\times\hat{\boldsymbol{k}}|$, z, and ζ are independent variables. Here ζ is the azimuthal angle of the normal to the lepton plane, so that

$$[\hat{\boldsymbol{q}}\,.\,(\hat{\boldsymbol{d}}\times\hat{\boldsymbol{k}})]^2 = (1-z^2)|\hat{\boldsymbol{d}}\times\hat{\boldsymbol{k}}|^2\cos^2\zeta. \tag{A.5}$$

Therefore the minima of

$$-3z^2-3[\boldsymbol{q}\,.\,(\hat{\boldsymbol{d}}\times\hat{\boldsymbol{k}})]^2 \tag{A.6}$$

occur at (i) $z^2 = 1$, any value of $|\hat{\boldsymbol{d}}\times\hat{\boldsymbol{k}}|$ and ζ, and (ii) $|\hat{\boldsymbol{d}}\times\hat{\boldsymbol{k}}| = \cos\zeta = 1$ and any value of z. Rewriting the other terms in the curly bracket of (A.1) as

$$\tfrac{1}{2}\{11-3z+(5-3z)[1-(\hat{\boldsymbol{d}}\times\hat{\boldsymbol{k}})^2]\}, \tag{A.7}$$

it is easy to see that the minimum occurs at $z = |\hat{\boldsymbol{d}}\times\hat{\boldsymbol{k}}| = 1$.

Combining the remarks concerning (A.6) and (A.7), we see that

$$(\mathscr{F}_\gamma)_{\min} \sim (0{\cdot}2)N \tag{A.8}$$

occurring at

$$z = |\hat{\boldsymbol{d}}\times\hat{\boldsymbol{k}}| = 1. \tag{A.9}$$

Note that, at $z = 1$, $\cos\zeta = 1$ automatically.

(A.3) and (A.8) give

$$\max(\mathscr{F}_\pi/\mathscr{F}_\gamma) \sim 10,$$

since condition (A.9) is compatible with (A.4).

References

ABBHHM collaboration 1968 *Nucl. Phys.* B **8** 535
Baker M and Glashow S L 1962 *Nuovo Cim.* **25** 857
Beg M A B 1963 *Phys. Rev.* **132** 426
Berman S M and Geffen D A 1960 *Nuovo Cim.* **18** 1192
Cabibbo N and Ferrari E 1960 *Nuovo Cim.* **18** 928
Carithers W C, Modis T, Nygren D R, Pun T P, Schwartz E L, Sticker H, Steinberger J, Weilhammer P and Christenson J H 1973 *Phys. Rev. Lett.* **30** 1336
Clark A R, Elioff T, Field R C, Firsch H J, Johnson R P, Kerth L T and Wenzel W A 1971 *Phys. Rev. Lett.* **26** 1667
Clegg A B 1969 *Proc. 4th Int. Symp. on Electron and Photon Interactions at High Energies, Daresbury* (Daresbury: Nuclear Physics Laboratory) p 123
Dalitz R H 1955 *Phys. Rev.* **99** 915
Dance D R 1971 *Cambridge Report* HEP/T/15

Donnachie A 1971 *Hadronic Interactions of Electrons and Photons* ed J Cumming and H Osborn (London: Academic Press)

Drell S D 1959 *Nuovo Cim.* **11** 692

Gjesdal S, Presser G, Steffen P, Steinberger J, Vannucci F, Wahl H, Filthuth H, Kleinknecht K, Lüth V and Zech G 1973 *Phys. Lett.* **44B** 217

Graham R H and Pakavasa S 1965 *Phys. Rev.* **140** B1144

Hyams B D, Kock W, Potter D C, von Linden L, Lorentz E, Lutjens G, Stierlin U and Weilhammer P 1969 *Phys. Lett.* **29B** 128

Källen G 1964 *Elementary Particle Physics* (London: Addison-Wesley) p 87

Kroll N M and Wada W 1955 *Phys. Rev.* **98** 1355

Litskevitch I K 1970 *Yadern. Fiz.* **10** 815 (1970 *Sov. J. nucl. Phys.* **10** 471)

Litskevitch I K and Franke V A 1970 *Yadern. Fiz.* **11** 1078 (1970 *Sov. J. nucl. Phys.* **11** 599)

Marshak R E, Riazuddin and Ryan C P 1969 *Theory of Weak Interactions in Particle Physics* (New York: Wiley, Interscience)

Particle Data Group 1973 *Rev. mod. Phys.* **45** 60

Pratrap M and Smith J 1972 *Phys. Rev.* D **5** 2020

Quigg C and Jackson J D 1968 *UCRL Report* 18487

Wehmann A W, Engels E, Hoffman C M, Innocenti P G, Wilson R, Blaupied W A, Drickey D J, Hand L H and Stairs D G 1968 *Phys. Rev. Lett.* **20** 748